

Smooth actions of compact quantum groups on compact smooth manifolds

Debashish Goswami¹ and Soumalya Joardar²

Indian Statistical Institute

203, B. T. Road, Kolkata 700108

Email: goswamid@isical.ac.in,

Phone: 0091 33 25753420, Fax: 0091 33 25773071

Abstract

Definition of a smooth action of a CQG on a compact, smooth manifold is given and studied. It is shown that a smooth action is always injective. Furthermore A necessary and sufficient condition for a lift of the smooth action as a bimodule morphism on the bimodule of one forms has been deduced and it is also shown to be equivalent to the condition of preserving some Riemannian inner product on the manifold.

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1 Introduction

It is a very important and interesting problem in the theory of quantum groups and noncommutative geometry to study ‘quantum symmetries’ of various classical and quantum structures. Indeed, symmetries of physical systems (classical or quantum) were conventionally modeled by group actions, and after the advent of quantum groups, group symmetries were naturally generalized to symmetries given by quantum group action. In the framework of Connes’ non commutative geometry it is natural to consider actions of compact quantum groups on spectral triples. In the topological setting the action of a compact quantum group (CQG for short) on a spectral triple was defined in [5] as the C^* action of the compact quantum group on the natural C^* algebra in the spectral triple. For the classical case it is nothing but a continuous action of the CQG on $C(M)$ where M is a compact manifold. It is worth mentioning that C^* action of a CQG on a C^* algebra has been extensively studied in [13], [15] and in other places. But in the context of noncommutative geometry the ‘space’ has some additional structures and thus one expects to go beyond the C^* action category. As in the group case, we should be able to talk about ‘smooth’ action of a CQG. In this paper one of our jobs has been to define and study a ‘smooth’ action of a CQG on a classical spectral triple, i.e. we consider the topological action of a CQG on the smooth algebra $C^\infty(M)$, where the algebra is endowed with its canonical Fréchet topology coming from derivations. We proved an interesting result about injectivity of a smooth action.

For a smooth action of a group on a compact smooth manifold, the ‘differential’ of the action automatically lifts as a well defined bimodule morphism to the space of one forms of the manifold. But in case of CQG it turns out that this lift is not automatic due to a fundamental noncommutativity. In fact, we have example of Hopf-algebra (of non compact type) having coaction on a coordinate algebra of an algebraic variety which does not admit such a lift. We give a necessary and sufficient condition for such a lift to exist. However, no such example is there yet with a smooth action of a compact quantum group.

We show that a smooth action is inner product preserving with respect to some Riemannian metric on the manifold if and only if it admits a lift $\beta : \Omega^1(C^\infty(M)) \rightarrow \Omega^1(C^\infty(M)) \hat{\otimes} \mathcal{Q}$ to the bimodule of one forms. Already the lift played a crucial role in studying the isometric action of a CQG on a compact, connected, Riemannian manifold. We believe that in the context of

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noncommutative geometry ([4]) such lifts are going to be important when we are going to treat CQG as ‘symmetry’ objects. More specifically, when we are going to consider invariance of various action functionals (Yang-Mills, Einstein-Hilbert) under CQG actions, such lifts might be very important.

2 Preliminaries

In this paper all the Hilbert spaces are over \mathbb{C} unless mentioned otherwise. If V is a vector space over real numbers we denote its complexification by $V_{\mathbb{C}}$. For a vector space V , V' stands for its algebraic dual. \oplus and \otimes will denote the algebraic direct sum and algebraic tensor product respectively. We shall denote the C^* algebra of bounded operators on a Hilbert space \mathcal{H} by $\mathcal{B}(\mathcal{H})$ and the C^* algebra of compact operators on \mathcal{H} by $\mathcal{B}_0(\mathcal{H})$. Sp, \overline{Sp} stand for the linear span and closed linear span of elements of a vector space respectively, whereas $\text{Im}(A)$ denotes the image of a linear map. We denote by WOT and SOT the weak operator topology and the strong operator topology respectively. Let \mathcal{C} be an algebra. Then $\sigma_{ij} : \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}} \rightarrow \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}}$

is the flip map between i and j -th place and $m_{ij} : \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{n\text{-times}} \rightarrow \underbrace{\mathcal{C} \otimes \mathcal{C} \otimes \dots \otimes \mathcal{C}}_{(n-1)\text{-times}}$ is the map

obtained by multiplying i and j -th entry. In case we have two copies of an algebra we shall simply denote by σ and m for the flip and multiplication map respectively.

2.1 Locally convex $*$ algebras and their tensor products

We begin by recalling from [16] the tensor product of two C^* algebras \mathcal{C}_1 and \mathcal{C}_2 and let us choose the minimal or spatial tensor products between two C^* algebras. The corresponding C^* algebra will be denoted by $\mathcal{C}_1 \hat{\otimes} \mathcal{C}_2$ throughout this paper. However we need to consider more general topological spaces and algebras. A locally convex space is a vector space equipped with a locally convex topology given by a family of seminorms. We call a locally convex space Fréchet if the family of seminorms is countable (hence the space is metrizable) and is complete with respect to the metric given by the family of seminorms. There are many ways to equip the algebraic tensor product of two locally convex spaces with a locally convex topology. Let E_1, E_2 be two locally convex spaces with family of seminorms $\{\|\cdot\|_{1,i}\}$ and $\{\|\cdot\|_{2,j}\}$ respectively. Then one wants a family $\{\|\cdot\|_{i,j}\}$ of seminorms for $E_1 \otimes E_2$ such that $\|e_1 \otimes e_2\|_{i,j} = \|e_1\|_{1,i} \|e_2\|_{2,j}$. The problem is that such a choice is far from unique and there is a maximal and a minimal choice giving the projective and injective tensor product respectively. A Fréchet locally convex space is called nuclear if its projective and injective tensor products with any other Fréchet space coincide as a locally convex space. We do not go into further details of this topic here but refer the reader to [17] for a comprehensive discussion. Furthermore if the space is a $*$ algebra then we demand that its $*$ algebraic structure is compatible with its locally convex topology i.e. the involution $*$ is continuous and multiplication is jointly continuous with respect to the topology. Projective and injective tensor product of two such topological $*$ algebras are again topological $*$ algebra. We shall mostly use unital $*$ algebras. Henceforth all the topological $*$ -algebras will be unital unless otherwise mentioned.

We now specialize to a particular class of locally convex $*$ algebras called smooth C^* -normed algebras defined and studied by Blackadar and Cuntz in [3]. These are C^* -normed $*$ -algebras which are complete w.r.t. the locally convex topology given by all closable derived seminorms (in the sense of [3]). It is proved in [3] that such algebras are closed under holomorphic functional calculus, and embedded as a norm-dense $*$ -subalgebra in a (unique upto isomorphism) C^* algebra. Moreover, any unital $*$ -homomorphism between such algebras is automatically continuous w.r.t. the corresponding locally convex topologies.

We actually need a slightly smaller subclass of such algebras, to be called ‘nice algebra’ for

convenience. This is one of special class of examples of Fréchet (D_∞^*) -subalgebra of C^* algebra treated in [2].

Definition 2.1 *A unital Fréchet $*$ algebra \mathcal{A} will be called a ‘nice’ algebra if there is a C^* -norm $\|\cdot\|$ on \mathcal{A} and the underlying locally convex topology of \mathcal{A} comes from the family of seminorms $\|\cdot\|_{\underline{\alpha}}$, with $\underline{\alpha} = (i_1, \dots, i_k)$ being any finite multi index (including the empty index, i.e. $\alpha = \phi$) and where*

$$\|x\|_{\underline{\alpha}} := \|\delta_{\underline{\alpha}}(x)\| \equiv \|\delta_{i_1} \dots \delta_{i_k}(x)\|,$$

$\delta_\phi := \text{id}$ and each δ_i denotes a $\|\cdot\|$ -closable $*$ -derivation from \mathcal{A} to itself.

Given two such ‘nice’ algebras $\mathcal{A}(\subset \mathcal{A}_1)$ and $\mathcal{B}(\subset \mathcal{B}_1)$, where $\mathcal{A}_1, \mathcal{B}_1$ denote respectively the C^* -completion of \mathcal{A}, \mathcal{B} in the corresponding C^* -norms, we choose the injective tensor product norm on $\mathcal{A} \otimes \mathcal{B}$, i.e. we view it as a dense subalgebra of $\mathcal{A}_1 \hat{\otimes} \mathcal{B}_1$. $\mathcal{A} \otimes \mathcal{B}$ has natural (closable $*$) derivations of the forms $\tilde{\delta} = \delta \otimes \text{id}$ as well as $\tilde{\eta} = \text{id} \otimes \eta$ where δ, η are closable $*$ -derivations on \mathcal{A} and \mathcal{B} respectively. Clearly, $\tilde{\delta}$ commute with $\tilde{\eta}$. We topologize $\mathcal{A} \otimes \mathcal{B}$ by the family of seminorms coming from such derivations, i.e. $\{\|\cdot\|_{\underline{\alpha}\underline{\beta}}\}$ where $\|\cdot\|$ is the injective C^* -norm and

$$\|X\|_{\underline{\alpha}\underline{\beta}} = \|\tilde{\delta}_{\underline{\alpha}}\tilde{\eta}_{\underline{\beta}}(X)\|,$$

$\underline{\alpha} = (i_1, \dots, i_k)$, $\underline{\beta} = (j_1, \dots, j_l)$ some multi indices as before and δ_i, η_j ’s being closable $*$ -derivations on \mathcal{A} and \mathcal{B} respectively.

We denote the completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to this topology by $\mathcal{A} \hat{\otimes} \mathcal{B}$. It follows from [3] that it is indeed a C^* -normed smooth algebra and it is also Fréchet. However, we cannot in general argue that it is nice in our sense as there may be more derivations on the completion which do not ‘split’ as a sum of derivations on the two constituent algebras. Fortunately, all the locally convex $*$ algebras considered in this paper will be of the form $C^\infty(M) \hat{\otimes} \mathcal{Q}$ for some compact manifold M and some unital C^* algebra \mathcal{Q} so that they will be ‘nice’ locally convex $*$ algebras in our sense with $C(M) \hat{\otimes} \mathcal{Q}$ as the ambient C^* algebra and the finitely many canonical derivations coming from the coordinate vector fields of the compact manifold M . We will show in one of the appendices that the above tensor product $\hat{\otimes}$ of such algebras will turn out to be nice again.

In case a nice algebra \mathcal{B} is complete in the C^* -norm, i.e. a C^* -algebra, all its $*$ -derivations are norm-bounded. Thus, for any other nice algebra \mathcal{A} , the topology of the tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$ is determined by $\tilde{\delta}$ ’s only, where δ ’s are closable $*$ -derivations on \mathcal{A} .

Given nice $*$ algebras $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ and $u : \mathcal{E}_1 \rightarrow \mathcal{E}_2, v : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ two unital $*$ -homomorphisms which are automatically continuous, the algebraic tensor product map $u \otimes v$ can be shown to be continuous with respect to the locally convex topology discussed above. We denote the continuous extension again by $u \otimes v$. Using this, we can also define $(\text{id} \otimes \omega) : \mathcal{E}_1 \hat{\otimes} \mathcal{E}_2 \rightarrow \mathcal{E}_1$ if \mathcal{E}_2 is a C^* algebra and ω is any state, or more generally, a bounded linear functional.

We note the following standard fact without proof which will be crucial in the analysis of smooth actions of compact quantum groups later on.

Proposition 2.2 *If $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are nice algebras as above and $\Phi : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathcal{A}_3$ is a bilinear map which is separately continuous in each of the arguments. Then Φ extends to a continuous linear map from the projective tensor product of \mathcal{A}_1 with \mathcal{A}_2 to \mathcal{A}_3 . If furthermore, \mathcal{A}_1 is nuclear, Φ extends to a continuous map from $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ to \mathcal{A}_3 .*

As a special case, suppose that $\mathcal{A}_1, \mathcal{A}_2$ are subalgebras of a nice algebra \mathcal{A} and also that \mathcal{A}_1 is isomorphic as a Fréchet space to some nuclear space. Then the multiplication map, say m , of \mathcal{A} extends to a continuous map from $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ to \mathcal{A} .

2.2 Exterior tensor product of Hilbert bimodules

Let \mathcal{E}_1 and \mathcal{E}_2 be two pre Hilbert bimodules over two locally convex $*$ algebras \mathcal{C}_1 and \mathcal{C}_2 which are subalgebras of nice algebras respectively. We denote the algebra valued inner product for

the pre Hilbert bimodules by $\langle\langle, \rangle\rangle$. When the bimodule is a pre Hilbert space, we denote the corresponding scalar valued inner product by \langle, \rangle . Then $\mathcal{E}_1 \otimes \mathcal{E}_2$ has an obvious $\mathcal{C}_1 \otimes \mathcal{C}_2$ bimodule structure, given by $(a \otimes b)(e_1 \otimes e_2)(a' \otimes b') = ae_1a' \otimes be_2b'$ for $a, a' \in \mathcal{C}_1, b, b' \in \mathcal{C}_2$ and $e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2$. Also define $\mathcal{C}_1 \otimes \mathcal{C}_2$ valued inner product by $\langle\langle e_1 \otimes e_2, f_1 \otimes f_2 \rangle\rangle = \langle\langle e_1, f_1 \rangle\rangle \otimes \langle\langle e_2, f_2 \rangle\rangle$ for $e_1, f_1 \in \mathcal{E}_1$ and $e_2, f_2 \in \mathcal{E}_2$. In case $\mathcal{C}_1, \mathcal{C}_2$ are complete, i.e. nice algebras and $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert bimodules, we denote the completed module by $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$. In fact $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$ is an $\mathcal{C}_1 \bar{\otimes} \mathcal{C}_2$ bimodule. This is called the exterior tensor product of two bimodules. In particular if one of the bimodule is a Hilbert space \mathcal{H} (bimodule over \mathbb{C}) and the other is a C^* algebra \mathcal{Q} (bimodule over itself), then the exterior tensor product gives the usual Hilbert \mathcal{Q} module $\mathcal{H} \bar{\otimes} \mathcal{Q}$. When $\mathcal{H} = \mathbb{C}^N$, we have a natural identification of an element $T = ((T_{ij})) \in M_N(\mathcal{Q})$ with the right \mathcal{Q} linear map of $\mathbb{C}^N \otimes \mathcal{Q}$ given by

$$e_i \mapsto \sum e_j \otimes T_{ji},$$

where $\{e_i\}_{i=1, \dots, N}$ is a basis for \mathbb{C}^N . We shall need tensor product of maps, which are ‘isometric’ in some sense. Let $T_i : \mathcal{E}_i \rightarrow \mathcal{F}_i, i = 1, 2$ be two \mathbb{C} -linear maps and $\mathcal{E}_i, \mathcal{F}_i$ be Hilbert bimodules over $\mathcal{C}_i, \mathcal{D}_i$ ($i = 1, 2$) respectively. Moreover, suppose that $\langle\langle T_i(\xi_i), T_i(\eta_i) \rangle\rangle = \alpha_i \langle\langle \xi_i, \eta_i \rangle\rangle, \xi_i, \eta_i \in \mathcal{E}_i$ where $\alpha_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$ are $*$ -homomorphisms. Then it is easy to show that the algebraic tensor product $T := T_1 \otimes_{alg} T_2$ also satisfies $\langle\langle T(\xi), T(\eta) \rangle\rangle = (\alpha_1 \otimes \alpha_2)(\langle\langle \xi, \eta \rangle\rangle)$ and hence extends to a well defined continuous map from $\mathcal{E}_1 \bar{\otimes} \mathcal{E}_2$ to $\mathcal{F}_1 \bar{\otimes} \mathcal{F}_2$ again to be denoted by $T_1 \otimes T_2$.

2.3 Compact quantum groups, their representations and actions

2.3.1 Definition and representations of compact quantum groups

A compact quantum group (CQG for short) is a unital C^* algebra \mathcal{Q} with a coassociative coproduct (see [11], [20]) Δ from \mathcal{Q} to $\mathcal{Q} \hat{\otimes} \mathcal{Q}$ such that each of the linear spans of $\Delta(\mathcal{Q})(\mathcal{Q} \otimes 1)$ and that of $\Delta(\mathcal{Q})(1 \otimes \mathcal{Q})$ is norm-dense in $\mathcal{Q} \hat{\otimes} \mathcal{Q}$. From this condition, one can obtain a canonical dense unital $*$ -subalgebra \mathcal{Q}_0 of \mathcal{Q} on which linear maps κ and ϵ (called the antipode and the counit respectively) are defined making the above subalgebra a Hopf $*$ -algebra. In fact, this is the algebra generated by the ‘matrix coefficients’ of the (finite dimensional) irreducible non degenerate representations (to be defined shortly) of the CQG. The antipode is an anti-homomorphism and also satisfies $\kappa(a^*) = (\kappa^{-1}(a))^*$ for $a \in \mathcal{Q}_0$.

It is known that there is a unique state h on a CQG \mathcal{Q} (called the Haar state) which is bi-invariant in the sense that $(\text{id} \otimes h) \circ \Delta(a) = (h \otimes \text{id}) \circ \Delta(a) = h(a)1$ for all a . The Haar state need not be faithful in general, though it is always faithful on \mathcal{Q}_0 at least. Given the Hopf $*$ -algebra \mathcal{Q}_0 , there can be several CQG’s which have this $*$ -algebra as the Hopf $*$ -algebra generated by the matrix elements of finite dimensional representations. We need two of such CQG’s: the reduced and the universal one. By definition, the reduced CQG \mathcal{Q}_r is the image of \mathcal{Q} in the GNS representation of h , i.e. $\mathcal{Q}_r = \pi_r(\mathcal{Q}), \pi_r : \mathcal{Q} \rightarrow \mathcal{B}(L^2(h))$ is the GNS representation.

There also exists a largest such CQG \mathcal{Q}^u , called the universal CQG corresponding to \mathcal{Q}_0 . It is obtained as the universal enveloping C^* algebra of \mathcal{Q}_0 . We also say that a CQG \mathcal{Q} is universal if $\mathcal{Q} = \mathcal{Q}^u$. Given two CQG’s $(\mathcal{Q}_1, \Delta_1)$ and $(\mathcal{Q}_2, \Delta_2)$, a $*$ homomorphism $\pi : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is said to be a CQG morphism if $(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi$ on \mathcal{Q}_1 . In case π is surjective, \mathcal{Q}_2 is said to be a quantum subgroup of \mathcal{Q}_1 (or a Woronowicz subalgebra) and denoted by $\mathcal{Q}_2 \leq \mathcal{Q}_1$.

Let \mathcal{H} be a Hilbert space. Consider the multiplier algebra $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$. This algebra has two natural embeddings into $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q} \hat{\otimes} \mathcal{Q})$. The first one is obtained by extending the map $x \mapsto x \otimes 1$. The second one is obtained by composing this map with the flip on the last two factors. We will write w^{12} and w^{13} for the images of an element $w \in \mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ by these two maps respectively. Note that if \mathcal{H} is finite dimensional then $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{Q})$ is isomorphic to $\mathcal{B}(\mathcal{H}) \otimes \mathcal{Q}$ (we do not need any topological completion).

Definition 2.3 Let (\mathcal{Q}, Δ) be a CQG. A unitary representation of \mathcal{Q} on a Hilbert space \mathcal{H} is a \mathbb{C} -linear map U from \mathcal{H} to the Hilbert module $\mathcal{H} \bar{\otimes} \mathcal{Q}$ such that

1. $\langle \langle U(\xi), U(\eta) \rangle \rangle = \langle \xi, \eta \rangle 1_{\mathcal{Q}}$, where $\xi, \eta \in \mathcal{H}$.
2. $(U \otimes \text{id})U = (\text{id} \otimes \Delta)U$.

Given such a unitary representation we have a unitary element \tilde{U} belonging to $\mathcal{M}(\mathcal{B}_0(\mathcal{H}) \hat{\otimes} \mathcal{S})$ given by $\tilde{U}(\xi \otimes b) = U(\xi)b$, $(\xi \in \mathcal{H}, b \in \mathcal{S})$ satisfying $(\text{id} \otimes \Delta)\tilde{U} = \tilde{U}^{12}\tilde{U}^{13}$, where $\mathcal{M}(\mathcal{C})$ denotes the multiplier algebra of a C^* algebra \mathcal{C} .

Definition 2.4 A closed subspace \mathcal{H}_1 of \mathcal{H} is said to be invariant if $U(\mathcal{H}_1) \subset \mathcal{H}_1 \bar{\otimes} \mathcal{Q}$. A unitary representation U of a CQG is said to be irreducible if there is no proper invariant subspace.

It is a well known fact that every irreducible unitary representation is finite dimensional.

We denote by $\text{Rep}(\mathcal{Q})$ the set of inequivalent irreducible unitary representations of \mathcal{Q} . For $\pi \in \text{Rep}(\mathcal{Q})$, let d_π and $\{t_{jk}^\pi : j, k = 1, \dots, d_\pi\}$ be the dimension and matrix coefficients of the corresponding finite dimensional representation respectively. Then for each $\pi \in \text{Rep}(\mathcal{Q})$, we have a unique $d_\pi \times d_\pi$ complex matrix F_π such that

- (1) F_π is positive and invertible with $\text{Tr}(F_\pi) = \text{Tr}(F_\pi^{-1}) = M_\pi > 0$ (say).
- (2) $h(t_{ij}^\pi, t_{kl}^{\pi*}) = \frac{1}{M_\pi} \delta_{ik} F_\pi(j, l)$.

Corresponding to $\pi \in \text{Rep}(\mathcal{Q})$, let ρ_{sm}^π be the linear functional on \mathcal{Q} given by $\rho_{sm}^\pi(x) = h(x_{sm}^\pi, x)$, $s, m = 1, \dots, d_\pi$ for $x \in \mathcal{Q}$ where $x_{sm}^\pi = (M_\pi) t_{km}^{\pi*} (F_\pi)_{ks}$. Also let $\rho^\pi = \sum_{s=1}^{d_\pi} \rho_{ss}^\pi$.

We say a map $\Gamma : \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{Q}_0$ (where \mathcal{K} is a vector space apriori without any topology) is an algebraic representation of the CQG \mathcal{Q} if Γ is algebraic, $(\Gamma \otimes \text{id})\Gamma = (\text{id} \otimes \Delta)\Gamma$ and $\text{Sp } \Gamma(\mathcal{K})\mathcal{Q}_0 = \mathcal{K} \otimes \mathcal{Q}_0$.

2.3.2 Actions of compact quantum groups

Let \mathcal{C} be a nice unital Fréchet $*$ -algebra (in the sense discussed in Subsection 3.1) and \mathcal{Q} be a compact quantum group.

Definition 2.5 A \mathbb{C} linear map $\alpha : \mathcal{C} \rightarrow \mathcal{C} \hat{\otimes} \mathcal{Q}$ is said to be a topological action of \mathcal{Q} on \mathcal{C} if

1. α is a continuous $*$ algebra homomorphism.
2. $(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha$ (co-associativity).
3. $\text{Sp } \alpha(\mathcal{C})(1 \otimes \mathcal{Q})$ is dense in $\mathcal{C} \hat{\otimes} \mathcal{Q}$ in the corresponding Fréchet topology.

Note that if the Fréchet algebra is a C^* algebra, then the definition of a topological action coincides with the usual C^* action of a compact quantum group.

Definition 2.6 A topological action α is said to be faithful if the $*$ -subalgebra of \mathcal{Q} generated by the elements of the form $(\omega \otimes \text{id})\alpha$, where ω is a continuous linear functional on \mathcal{C} , is dense in \mathcal{Q} .

Definition 2.7 Let $(H, \Delta, \epsilon, \kappa)$ be a Hopf $*$ algebra and A be a $*$ algebra. A unital $*$ algebra homomorphism $\alpha : A \rightarrow A \otimes H$ is said to be a Hopf $*$ algebraic (co)action of H on A if

- (i) $(\alpha \otimes \text{id})\alpha = (\text{id} \otimes \Delta)\alpha$.
- (ii) $(\text{id} \otimes \epsilon)\alpha = \text{id}$.

For a Hopf $*$ algebraic (co)action as above, one can prove that $\text{Sp } \{\alpha(A)(1 \otimes H)\} = A \otimes H$.

Remark 2.8 For a topological action α of the CQG \mathcal{Q} on the nice algebra \mathcal{C} , we say that it is algebraic over a $*$ -subalgebra $\mathcal{C}_0 \subset \mathcal{C}$ if $\alpha|_{\mathcal{C}_0} : \mathcal{C}_0 \rightarrow \mathcal{C}_0 \otimes \mathcal{Q}_0$ is a Hopf $*$ -algebraic action.

Let X be a compact space. Then we can consider the C^* action of \mathcal{Q} on $C(X)$. We say \mathcal{Q} acts topologically on a compact space X if there is a C^* action of \mathcal{Q} on $C(X)$. We have the following([8]):

Proposition 2.9 *If a CQG \mathcal{Q} acts topologically and faithfully on X , where X is any compact space, then the corresponding reduced CQG \mathcal{Q}_r (which has a faithful Haar state) must be a Kac algebra. In particular the Haar state of \mathcal{Q}_r , and hence of \mathcal{Q} is tracial. Moreover, the antipode κ is defined and norm-bounded on \mathcal{Q}_r .*

Now recall from Subsection 4.1 the linear functional ρ^π on a compact quantum group \mathcal{Q} . Given a topological action α of \mathcal{Q} on a ‘nice’ unital $*$ algebra \mathcal{C} , we can define a projection $P_\pi : \mathcal{C} \rightarrow \mathcal{C}$ called spectral projection corresponding to $\pi \in \text{Rep}(\mathcal{Q})$ by $P_\pi := (\text{id} \otimes \rho^\pi)\alpha$. Note that $(\text{id} \otimes \phi)\alpha(\mathcal{C}) \subset \mathcal{C}$ for all bounded linear functionals ϕ on \mathcal{Q} . We call $\mathcal{C}_\pi := \text{Im } P_\pi$ the spectral subspace corresponding to π . The subspace spanned by $\{\mathcal{C}_\pi, \pi \in \text{Rep}(\mathcal{Q})\}$ is actually a unital $*$ -subalgebra called the spectral subalgebra for the action. Let $\mathcal{C}_0 := \ker(\alpha) \oplus \text{Sp}\{\mathcal{C}_\pi : \pi \in \hat{\mathcal{Q}}\}$. So in particular if $\ker(\alpha) = \{0\}$, then the spectral subspace coincides with \mathcal{C}_0 . Along the lines of [13] and Proposition 2.2 of [15] we have

Proposition 2.10 *(i) \mathcal{C}_0 is a unital $*$ subalgebra over which α is algebraic.*

(ii) \mathcal{C}_0 is dense in \mathcal{C} in the Fréchet topology.

(iii) \mathcal{C}_0 is maximal among the subspaces $V \subset \mathcal{C}$ with the property that $\alpha(V) \subset V \otimes \mathcal{Q}_0$.

We end this subsection with a discussion on an analogue of an action on a von Neumann algebra given by conjugating unitary representation. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and U be a unitary representation of a CQG \mathcal{Q} on \mathcal{H} . The map $\text{ad}_{\tilde{U}}$ is a normal, injective $*$ -homomorphism on $\mathcal{B}(\mathcal{H})$. We say $\text{ad}_{\tilde{U}}$ leaves \mathcal{M} invariant if $(\text{id} \otimes \phi)(\text{ad}_{\tilde{U}}(x)) \in \mathcal{M}$ for all bounded linear functionals ϕ on \mathcal{Q} . In that case we can consider the spectral projections $P_\pi = (\text{id} \otimes \rho^\pi) \circ \text{ad}_{\tilde{U}}$ and define $\mathcal{M}_\pi := P_\pi(\mathcal{M})$. As before, we have the spectral subalgebra $\mathcal{M}_0 = \text{Sp}\{\mathcal{M}_\pi, \pi \in \hat{\mathcal{Q}}\}$ and as $\text{ad}_{\tilde{U}}$ is one to one, \mathcal{M}_0 is maximal among subspaces $V \subset \mathcal{M}$ for which $\text{ad}_{\tilde{U}}(V) \subset V \otimes \mathcal{Q}_0$. Furthermore, it can be proved that \mathcal{M}_0 is SOT dense in \mathcal{M} .

We end this subsection with a discussion on the unitary implementability of an action.

Definition 2.11 *We call an action α of a CQG \mathcal{Q} on a unital C^* algebra \mathcal{C} to be implemented by a unitary representation U of \mathcal{Q} in \mathcal{H} , say, if there is a faithful representation $\pi : \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$ such that $\tilde{U}(\pi(x) \otimes 1)\tilde{U}^* = (\pi \otimes \text{id})(\alpha(x))$ for all $x \in \mathcal{C}$.*

It is clear that if an action is implemented by a unitary representation then it is one-to-one. In fact, as $(\text{id} \otimes \pi_r)(U)$ gives a unitary representation of \mathcal{Q}_r in \mathcal{H} and the ‘reduced action’ $\alpha_r := (\text{id} \otimes \pi_r) \circ \alpha$ of \mathcal{Q}_r is also implemented by a unitary representation, it follows that even α_r is one-to-one. We see below that this is actually equivalent to implementability by unitary representation:

Lemma 2.12 *Given an action α of \mathcal{Q} on a unital separable C^* algebra \mathcal{C} the following are equivalent:*

(a) There is a faithful positive functional ϕ on \mathcal{C} which is invariant w.r.t. α , i.e. $(\phi \otimes \text{id})(\alpha(x)) = \phi(x)1_{\mathcal{Q}}$ for all $x \in \mathcal{C}$.

(b) The action is implemented by some unitary representation.

(c) The reduced action α_r of \mathcal{Q}_r is injective.

Proof:

If (a) holds, we consider \mathcal{H} to be the GNS space of the faithful positive functional ϕ . The GNS representation π is faithful, and the linear map U defined by $U(x) := \alpha(x)$ from $\mathcal{C} \subset \mathcal{H} = L^2(\mathcal{C}, \phi)$ to $\mathcal{H} \bar{\otimes} \mathcal{Q}$ is an isometry by the invariance of ϕ . Thus U extends to \mathcal{H} and it is easy to check that it gives a unitary representation which implements α .

We have already argued $(b) \Rightarrow (c)$, and finally, if (c) holds, we choose any faithful state say τ on the separable C^* algebra \mathcal{C} and take $\phi(x) = (\tau \otimes h)(\alpha_r(x))$, which is faithful as h is faithful on \mathcal{Q}_r . It can easily be verified that ϕ is α -invariant on the dense subalgebra \mathcal{C}_0 mentioned before, and hence on the whole of \mathcal{C} . \square

2.4 Representation of CQG on a Hilbert bimodule over a nice topological $*$ -algebra

We now generalize the notion of unitary representation on Hilbert spaces in another direction, namely on Hilbert bimodules over nice, unital topological $*$ -algebras. Let \mathcal{E} be a Hilbert $\mathcal{C} - \mathcal{D}$ bimodule over topological $*$ -algebras \mathcal{C} and \mathcal{D} . Also let \mathcal{Q} be a compact quantum group. If we consider \mathcal{Q} as a bimodule over itself, then we can form the exterior tensor product $\mathcal{E} \bar{\otimes} \mathcal{Q}$ which is a $\mathcal{C} \hat{\otimes} \mathcal{Q} - \mathcal{D} \hat{\otimes} \mathcal{Q}$ bimodule. Also let $\alpha_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \hat{\otimes} \mathcal{Q}$ and $\alpha_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \hat{\otimes} \mathcal{Q}$ be topological actions on \mathcal{C} and \mathcal{D} of \mathcal{Q} in the sense discussed earlier. Using α we can give $\mathcal{E} \bar{\otimes} \mathcal{Q}$ a $\mathcal{C} - \mathcal{D}$ bimodule structure given by $a \cdot \eta \cdot a' = \alpha_{\mathcal{C}}(a) \eta \alpha_{\mathcal{D}}(a')$, for $\eta \in \mathcal{E} \bar{\otimes} \mathcal{Q}$ and $a \in \mathcal{C}, a' \in \mathcal{D}$ (but without any \mathcal{D} valued inner product).

Definition 2.13 A \mathbb{C} -linear map $\Gamma : \mathcal{E} \rightarrow \mathcal{E} \bar{\otimes} \mathcal{Q}$ is said to be an $\alpha_{\mathcal{D}}$ equivariant unitary representation of \mathcal{Q} on \mathcal{E} if

1. $\Gamma(\xi d) = \Gamma(\xi) \alpha_{\mathcal{D}}(d)$ and $\Gamma(c \xi) = \alpha_{\mathcal{C}}(c) \Gamma(\xi)$ for $c \in \mathcal{C}, d \in \mathcal{D}$.
2. $\langle \langle \Gamma(\xi), \Gamma(\xi') \rangle \rangle = \alpha_{\mathcal{D}}(\langle \langle \xi, \xi' \rangle \rangle)$, for $\xi, \xi' \in \mathcal{E}$.
3. $(\Gamma \otimes \text{id})\Gamma = (\text{id} \otimes \Delta)\Gamma$ (co associativity)
4. $\overline{Sp} \Gamma(\mathcal{E})(1 \otimes \mathcal{Q}) = \mathcal{E} \bar{\otimes} \mathcal{Q}$ (non degeneracy).

In the above definition note that condition (2) allows one to define $(\Gamma \otimes \text{id})$. If $\mathcal{C} = \mathcal{D}$ and $\alpha_{\mathcal{C}} = \alpha_{\mathcal{D}}$ we simply call Γ α -equivariant. If $\alpha_{\mathcal{C}}$ and $\alpha_{\mathcal{D}}$ are understood from the context, we may call Γ just equivariant. Given an α equivariant representation Γ of \mathcal{Q} on a Hilbert bimodule \mathcal{E} , proceeding as in Subsection 4.2, we define $P_{\pi} := (\text{id} \otimes \rho^{\pi}) \circ \Gamma$, $\mathcal{E}_{\pi} := \text{Im} P_{\pi}$ for $\pi \in \text{Rep}(\mathcal{Q})$ and $\mathcal{E}_0 := \overline{Sp} \{ \mathcal{E}_{\pi}; \pi \in \text{Rep}(\mathcal{Q}) \} \oplus \ker(\Gamma)$. In case Γ is one-one which is equivalent to $\alpha_{\mathcal{D}}$ being one-one, \mathcal{E}_0 coincides with $\overline{Sp} \{ \mathcal{E}_{\pi}; \pi \in \text{Rep}(\mathcal{Q}) \}$. Again proceeding along the lines of [13] and [15], we can prove the following analogue of Proposition 2.10:

Proposition 2.14 1. $(\text{id} \otimes \phi)\Gamma(\mathcal{E}) \subset \mathcal{E}$, for all bounded linear functional $\phi \in \mathcal{Q}^*$.

2. $P_{\pi}^2 = P_{\pi}$.
3. Γ is algebraic over \mathcal{E}_0 i.e. $\Gamma(\mathcal{E}_0) \subset \mathcal{E}_0 \otimes \mathcal{Q}_0$ and $(\text{id} \otimes \epsilon)\Gamma = \text{id}$ on \mathcal{E}_0 .
4. \mathcal{E}_0 is dense in \mathcal{E} .
5. \mathcal{E}_0 is maximal among the subspaces $V \subset \mathcal{E}$ such that $\Gamma(V) \subset V \otimes \mathcal{Q}$.

2.5 Locally convex $*$ algebras and Hilbert bimodules coming from classical geometry

2.5.1 $C^{\infty}(M)$ as a nice algebra

All the notations are as in the Subsection 3.1 and throughout the section, we denote a smooth n - dimensional compact manifold possibly with boundary by M . We denote the algebra of real (complex respectively) valued smooth functions on M by $C^{\infty}(M)_{\mathbb{R}}$ ($C^{\infty}(M)$ respectively). Clearly $C^{\infty}(M)$ is the complexification of $C^{\infty}(M)_{\mathbb{R}}$. We also equip it with a locally convex topology : we say a sequence $f_n \in C^{\infty}(M)$ converges to an $f \in C^{\infty}(M)$ if for every compact set K within a single coordinate neighborhood (M being compact, has finitely many such neighborhoods) and a multi-index α , $\partial^{\alpha} f_n \rightarrow \partial^{\alpha} f$ uniformly over K . Here, for a coordinate neighborhood (x_1, \dots, x_n) and a multi-index $\alpha = (i_1, \dots, i_k)$ with $i_j \in \{1, \dots, n\}$, ∂^{α} denotes $\frac{\partial}{\partial x_{i_1}} \dots \frac{\partial}{\partial x_{i_k}}$. Equivalently, let U_1, U_2, \dots, U_l be a finite cover of M . Then it is a locally convex topology described by a countable family of seminorms given by:

$$p_i^{K, \alpha}(f) = \sup_{x \in K} |\partial^{\alpha} f(x)|,$$

where K is a compact set within U_i , α is any multi index, $i = 1, 2, \dots, l$. $C^\infty(M)$ is complete with respect to this topology (Example 1.46 of [14] with obvious modifications) and hence this makes $C^\infty(M)$ a locally convex Fréchet $*$ algebra with obvious $*$ structure. In fact, it is nice algebra because any closable $*$ -derivation on $C^\infty(M)$ comes from a smooth vector field, i.e. locally a $C^\infty(M)$ -linear combination of partial derivatives.

Actually, by choosing a finite C^∞ partition of unity on the compact manifold M , we can obtain finite set $\{\delta_1, \dots, \delta_N\}$ for some $N \geq n$ of globally defined vector fields on M which is complete in the sense that $\{\delta_1(m), \dots, \delta_N(m)\}$ spans $T_m(M)$ for all m (need not be a basis). It follows that for describing the locally convex topology on $C^\infty(M)$ it is enough to consider the seminorms $\|\delta_{i_1} \dots \delta_{i_k}(\cdot)\|$, with $k \geq 0$, $i_j \in \{1, \dots, N\}$.

It is also known that (see Example 6.2 of [12])

Proposition 2.15 *With the above topology $C^\infty(M)$ is nuclear as a locally convex space.*

Let E be any locally convex space. Then we can define the space of E valued smooth functions on a compact manifold M . Take a centered coordinate chart (U, ψ) around a point $x \in M$. Then an E valued function f on M is said to be smooth at x if $f \circ \psi^{-1}$ is smooth E valued function at $0 \in \mathbb{R}^n$ in the sense of [17] (Definition 40.1). We denote the space of E valued smooth functions on M by $C^\infty(M, E)$. We can give a locally convex topology on $C^\infty(M, E)$ by the family of seminorms given by $p_i^{K, \alpha}(f) := \sup_{x \in K} \|\partial^\alpha f(x)\|$, where i, K, α are as before. Then we have the following

Proposition 2.16 *1. If E is complete, then so is $C^\infty(M, E)$.*

2. Suppose E is a complete locally convex space. Then we have $C^\infty(M) \hat{\otimes} E \cong C^\infty(M, E)$.

3. Let M and N be two smooth compact manifolds with boundary. Then $C^\infty(M) \hat{\otimes} C^\infty(N) \cong C^\infty(M, C^\infty(N)) \cong C^\infty(M \times N)$ and contains $C^\infty(M) \otimes C^\infty(N)$ as Fréchet dense subalgebra.

4. Let \mathcal{Q} be a C^ algebra. Then $C^\infty(M) \hat{\otimes} \mathcal{Q} \cong C^\infty(M, \mathcal{Q})$ as Fréchet algebras.*

For the proof of the first statement see 44.1 of [17]. The second statement also follows from 44.1 of [17] and the fact that $C^\infty(M)$ is nuclear. The third and fourth statements follow from the second statement (replacing E suitably).

Although nuclearity ensures that $C^\infty(M) \hat{\otimes} \mathcal{Q}$ is unambiguously defined and coincides with the tensor product for nice algebras defined earlier, it remains to see whether it is itself a nice algebra. Indeed, this is true and we'll give a proof in Appendix. In fact, we'll have more (see the Appendix):

Lemma 2.17 *For any two C^* algebras $\mathcal{Q}, \mathcal{Q}'$, $C^\infty(M, \mathcal{Q})$ is a nice algebra and the nice algebra tensor product $C^\infty(M, \mathcal{Q}) \hat{\otimes} \mathcal{Q}'$ is $*$ -isomorphic with $C^\infty(M) \hat{\otimes} (\mathcal{Q} \hat{\otimes} \mathcal{Q}') \cong C^\infty(M, \mathcal{Q} \hat{\otimes} \mathcal{Q}')$*

2.5.2 Hilbert bimodule of one-forms on a Riemannian manifold

Let $\Lambda^k(C^\infty(M))$ be the space of smooth k forms on the manifold M , with the natural locally convex topology induced by the topology of $C^\infty(M)$ given by a family of seminorms $q_i^{K, \alpha}(\omega) = \sup_{x \in K, 1 \leq j \leq n} |\partial^\alpha f_j(x)|$, $K \subset U_i$, where $K, \alpha, \{U_1, \dots, U_l\}$ are as before and $\omega|_{U_i} = \sum_{j=1}^n f_j dx_j$. It is clear from the definition that the differential map $d : C^\infty(M) \rightarrow \Omega^1(C^\infty(M))$ is Fréchet continuous.

Lemma 2.18 *Let \mathcal{A} be a Fréchet dense subalgebra of $C^\infty(M)$. Then $\Lambda^1(\mathcal{A}) := Sp \{fdg : f, g \in \mathcal{A}\}$ is dense in $\Lambda^1(C^\infty(M))$.*

Proof:

It is enough to approximate fdg where $f, g \in C^\infty(M)$ by elements of $\Lambda^1(\mathcal{A})$. By Fréchet density of \mathcal{A} in $C^\infty(M)$ we can choose sequences $f_m, g_m \in \mathcal{A}$ such that $f_m \rightarrow f$ and $g_m \rightarrow g$ in

the Fréchet topology, hence by the continuity of d and the $C^\infty(M)$ module multiplication of $\Lambda^1(C^\infty(M))$, we see that $f_m dg_m \rightarrow f dg$ in $\Lambda^1(C^\infty(M))$. \square

Let $\Omega^k(C^\infty(M))_u$ be the space of universal k -forms on the manifold M and δ be the derivation for the universal algebra of forms for $C^\infty(M)$ i.e $\delta : \Omega^k(C^\infty(M))_u \rightarrow \Omega^{k+1}(C^\infty(M))_u$ (see [10] for further details).

By the universal property \exists a surjective bimodule morphism $\pi \equiv \pi_{(1)} : \Omega^1(C^\infty(M))_u \rightarrow \Lambda^1(C^\infty(M))$, such that $\pi(\delta g) = dg$. $\Omega^1(C^\infty(M))_u$ has a $C^\infty(M)$ bimodule structure:

$$\begin{aligned} f\left(\sum_{i=1}^n g_i \delta h_i\right) &= \sum_{i=1}^n f g_i \delta h_i \\ \left(\sum_{i=1}^n g_i \delta h_i\right)f &= \sum_{i=1}^n (g_i \delta(h_i f) - g_i h_i \delta f) \end{aligned}$$

As M is compact, there is a Riemannian structure. Using the Riemannian structure on M we can equip $\Omega^1(C^\infty(M))$ with a $C^\infty(M)$ valued inner product $\langle\langle \sum_{i=1}^n f_i dg_i, \sum_{i=1}^n f'_i dg'_i \rangle\rangle \in C^\infty(M)$ by the following prescription:

for $x \in M$ choose a coordinate neighborhood $(U, x_1, x_2, \dots, x_n)$ around x such that dx_1, dx_2, \dots, dx_n is an orthonormal basis for T_x^*M . Note that the topology does not depend upon any particular choice of the Riemannian metric. Then

$$\langle\langle \sum_{i=1}^n f_i dg_i, \sum_{i=1}^n f'_i dg'_i \rangle\rangle (x) = \left(\sum_{i,j,k,l} \bar{f}_i f'_j \left(\frac{\partial \bar{g}_i}{\partial x_k} \frac{\partial g'_j}{\partial x_l} \right) \right) (x). \quad (1)$$

We see that a sequence $\omega_n \rightarrow \omega$ in $\Lambda^1(C^\infty(M))$ if $\langle\langle \omega_n - \omega, \omega_n - \omega \rangle\rangle \rightarrow 0$ in Fréchet topology of $C^\infty(M)$. With this $\Lambda^1(C^\infty(M))$ becomes a Hilbert module.

3 Smooth action of a CQG on a manifold

In this section we consider a compact manifold M possibly with boundary, but not necessarily orientable and discuss a notion of smoothness of a CQG action α of a CQG \mathcal{Q} . Moreover we prove that smoothness automatically implies injectivity of the reduced action, hence unitary implementability. This will follow from injectivity of the action on $C^\infty(M)$ which we will prove.

3.1 Definition of a smooth action

Definition 3.1 *A topological action (in the sense of Definition 2.5) of \mathcal{Q} on the Fréchet algebra $C^\infty(M)$ is called the smooth action of \mathcal{Q} on the manifold M . In case M has a boundary, we also assume that the closed ideal $\{f \in C^\infty(M) : f|_{\partial M} = 0\}$ is invariant by the action.*

Lemma 3.2 *A smooth action α of \mathcal{Q} on M extends to a C^* action on $C(M)$ which is denoted by α again.*

Proof:

It follows from generalities about smooth C^* -normed algebras proved in [3] (Proposition 6.8 there), which uses the fact that $C^\infty(M)$ is stable under taking square roots of positive invertible elements. \square

Lemma 3.3 *Given a C^* action $\alpha : C(M) \rightarrow C(M) \hat{\otimes} \mathcal{Q}$, $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$ if and only if $(\text{id} \otimes \phi)(\alpha(C^\infty(M))) \subset C^\infty(M)$ for all bounded linear functionals ϕ on \mathcal{Q} .*

Proof:

Only if part:

See discussion in Subsection 4.2.

If part:

follows from Corollary 5.5 of Appendix (Section 5). \square

Theorem 3.4 *Suppose we are given a C^* action α of \mathcal{Q} on M . Then following are equivalent:*

- 1) $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$ and $\text{Sp } \alpha(C^\infty(M))(1 \otimes \mathcal{Q})$ is Fréchet dense in $C^\infty(M, \mathcal{Q})$.
- 2) α is smooth.
- 3) $(\text{id} \otimes \phi)\alpha(C^\infty(M)) \subset C^\infty(M)$ for every state ϕ on \mathcal{Q} , and there is a Fréchet dense subalgebra \mathcal{A} of $C^\infty(M)$ over which α is algebraic.

Proof:

(1) \Rightarrow (2): Observe that it is enough to show that α is Fréchet continuous. But this follows from the Closed Graph Theorem, as Fréchet topology is stronger than the norm topology.

(2) \Rightarrow (3): Follows from the Proposition 2.10.

(3) \Rightarrow (1): From Lemma 3.3, it follows that $\alpha(C^\infty(M)) \subset C^\infty(M, \mathcal{Q})$. The density condition follows from densities of \mathcal{A} and $\mathcal{A} \otimes \mathcal{Q}_0$ in $C^\infty(M)$ and $C^\infty(M, \mathcal{Q})$ respectively. \square

3.2 Injectivity of the smooth action

In this subsection we show that for a smooth action $\alpha : C^\infty(M) \rightarrow C^\infty(M) \hat{\otimes} \mathcal{Q}$, the corresponding reduced C^* action on $C(M)$ is injective. Note that for a general C^* action this is not true. For a CQG \mathcal{Q} where \mathcal{Q} is a non amenable C^* algebra, the coproduct gives an action of the reduced CQG which is not injective (see [15]). We begin by proving an interesting fact which will be used later. For that recall that $C^\infty(M)$ is a nuclear locally convex space and hence so is any quotient by closed ideals.

Lemma 3.5 *If \mathcal{Q} has a faithful smooth action on $C^\infty(M)$, where M is compact manifold, then for every fixed $x \in M$ there is a well-defined, $*$ -homomorphic extension ϵ_x of the counit map ϵ to the unital $*$ -subalgebra $\mathcal{Q}_x^\infty := \{\alpha_r(f)(x) : f \in C^\infty(M)\}$ satisfying $\epsilon_x(\alpha(f)(x)) = f(x)$, where α_r is the reduced action discussed earlier.*

Proof:

Replacing \mathcal{Q} by \mathcal{Q}_r we can assume without loss of generality that \mathcal{Q} has faithful Haar state and $\alpha = \alpha_r$. In this case \mathcal{Q} will have bounded antipode κ (by Proposition 2.9). Let $\alpha_x : C^\infty(M) \rightarrow \mathcal{Q}_x^\infty$ be the map defined by $\alpha_x(f) := \alpha(f)(x)$. It is clearly continuous w.r.t. the Fréchet topology of $C^\infty(M)$ and hence the kernel say \mathcal{I}_x is a closed ideal, so that the quotient which is isomorphic to \mathcal{Q}_x^∞ is a nuclear space. Let us consider \mathcal{Q}_x^∞ with this topology and then by nuclearity, the projective and injective tensor products with \mathcal{Q} (viewed as a separable Banach space, where separability follows from the fact that \mathcal{Q} faithfully acts on the separable C^* algebra $C(M)$) coincide with $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q}$ and the multiplication map $m : \mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q} \rightarrow \mathcal{Q}$ is indeed continuous. Now, observe that $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q}$ is isomorphic as a Fréchet algebra with the quotient of $C^\infty(M) \hat{\otimes} \mathcal{Q}$ by the ideal $\text{Ker}(\alpha_x \otimes \text{id}) = \mathcal{I}_x \hat{\otimes} \mathcal{Q}$. Moreover, it follows from the relation $\Delta \circ \alpha = (\alpha \otimes \text{id}) \circ \alpha$ that Δ maps \mathcal{I}_x to $\mathcal{I}_x \hat{\otimes} \mathcal{Q}$, and in fact it is the restriction of the Fréchet-continuous map α there, hence induces a continuous map from $\mathcal{Q}_x^\infty \cong C^\infty(M)/\mathcal{I}_x$ to $\mathcal{Q}_x^\infty \hat{\otimes} \mathcal{Q} \cong (C^\infty(M) \hat{\otimes} \mathcal{Q})/(\mathcal{I}_x \hat{\otimes} \mathcal{Q})$. Thus, the composite map $m \circ (\text{id} \otimes \kappa) \circ \Delta : \mathcal{Q}_x^\infty \rightarrow \mathcal{Q}$ is continuous and this coincides with $\epsilon(\cdot)1_{\mathcal{Q}}$ on the Fréchet-dense subalgebra of \mathcal{Q}_x^∞ spanned by elements of the form $\alpha(f)(x)$, with f varying

in the Fréchet-dense spectral subalgebra of $C^\infty(M)$. This completes the proof of the lemma. \square

Corollary 3.6 *There is a well-defined extension of ϵ_x , say $\tilde{\epsilon}_x$, to the linear subspace spanned by elements of the form q_0q where $q \in \mathcal{Q}_x^\infty$ and $q_0 \in \mathcal{Q}_0$, given by $\tilde{\epsilon}_x(q_0q) = \epsilon(q_0)\epsilon_x(q)$. A similar conclusion will hold for subspace spanned by $\{qq_0, q \in \mathcal{Q}_x^\infty, q_0 \in \mathcal{Q}_0\}$.*

Proof:

We use the notation of Lemma 3.5. For a finite dimensional subspace \mathcal{D} of \mathcal{Q}_0 denote by \mathcal{D}_1 the subspace spanned by $(\text{id} \otimes \omega)(\Delta(\mathcal{D}))$ where ω varies over the (algebraic) dual of \mathcal{Q}_0 . Thus, $\Delta(\mathcal{D}) \subseteq \mathcal{D}_1 \otimes \mathcal{Q}_0$. Let $\mathcal{W} = \text{Span}(\mathcal{D}_1 \mathcal{Q}_x^\infty)$. As \mathcal{D}_1 is finite dimensional, $\mathcal{D}_1 \otimes \mathcal{Q}_x^\infty$ is nuclear and the multiplication map from $\mathcal{D}_1 \otimes \mathcal{Q}_x^\infty$ onto \mathcal{W} is continuous, hence \mathcal{W} can be viewed as a nuclear space in the quotient topology coming from $\mathcal{D}_1 \otimes \mathcal{Q}_x^\infty$. As in the proof of Lemma 3.5 we observe that $(\text{id} \otimes \kappa) \circ \Delta$ maps the subspace $\tilde{\mathcal{D}}$ spanned by $\mathcal{D}\mathcal{Q}_x^\infty$ to $\mathcal{W} \hat{\otimes} \mathcal{Q}$, hence $\epsilon_x^\mathcal{D}(\cdot)1 := m_{\mathcal{W}} \circ (\text{id} \otimes \kappa) \circ \Delta$ defines a continuous extension of the counit ϵ on $\tilde{\mathcal{D}}$ w.r.t. the (nuclear) quotient Fréchet topology of $\tilde{\mathcal{D}}$ coming from $\mathcal{D} \otimes \mathcal{Q}_x^\infty$. Here, $m_{\mathcal{W}}$ denotes the (continuous) multiplication map from $\mathcal{W} \hat{\otimes} \mathcal{Q}$ to \mathcal{Q} .

Moreover, $\tilde{\mathcal{D}} \cap \mathcal{Q}_0$ is dense in $\tilde{\mathcal{D}}$ as it contains elements of the form $q_0\alpha(f)(x)$ for $q \in \mathcal{D}$ and f in the (dense) spectral subalgebra of $C^\infty(M)$ and clearly, $\epsilon_x^\mathcal{D}$ agrees with ϵ (the original counit defined on \mathcal{Q}_0) on this dense subspace. Indeed, $\epsilon(q_0\alpha(f)(x)) = \epsilon(q_0)f(x) = \epsilon(q_0)\epsilon_x(\alpha(f)(x))$ for q_0, f as above, and hence by continuity, we can conclude that $\tilde{\epsilon}_x$ is the unique extension of ϵ to $\text{Span}(\mathcal{D}\mathcal{Q}_x^\infty)$. From the uniqueness, we see that $\epsilon_x^\mathcal{D}$ agrees with $\epsilon_x^\mathcal{U}$ on $\text{Span}((\mathcal{D} \cap \mathcal{U})\mathcal{Q}_x^\infty)$, for any two finite dimensional subspaces \mathcal{D}, \mathcal{U} of \mathcal{Q}_0 . In other words, the definition of $\epsilon_x^\mathcal{D}$ is independent of \mathcal{D} , and this gives a well-defined linear map $\tilde{\epsilon}_x$ on the whole of $\mathcal{Q}_0\mathcal{Q}_x^\infty$, which satisfies $\tilde{\epsilon}_x(q_0q) = \epsilon(q_0)\epsilon_x(q) = \epsilon(q_0)f(x)$ for all $q_0 \in \mathcal{Q}$ and $q = \alpha(f)(x)$ for f in the spectral subalgebra, hence by continuity for all $f \in C^\infty(M)$, completing the proof of the corollary.

For $\text{span}\mathcal{Q}_x^\infty\mathcal{Q}_0$ the proof is similar and hence omitted. \square

Corollary 3.7 *For any smooth action α on $C^\infty(M)$, the conditions of Theorem 2.12 are satisfied.*

Proof:

Replacing \mathcal{Q} by the Woronowicz subalgebra generated by $\{\alpha(f)(x), f \in C(M), x \in M\}$ we may assume that α is faithful. If $\alpha_r(f) = 0$ for $f \in C^\infty(M)$ then by Lemma 3.5 applying the extended ϵ we conclude $f = 0$. Now, consider any positive Borel measure μ of full support on M , with ϕ_μ being the positive functional obtained by integration w.r.t μ . Let $\psi := (\phi_\mu \otimes h) \circ \alpha_r$ be the positive functional which is clearly α_r -invariant and faithful on $C^\infty(M)$, i.e. $\psi(f) = 0, f \in C^\infty(M)$ and f nonnegative implies $f = 0$. But then by Riesz Representation Theorem there is a positive Borel measure ν such that $\psi(f) = \int_M f d\nu$. It follows that ν has full support, hence ψ is faithful also on $C(M)$. Indeed, for any nonempty open subset U of M there is a nonzero positive $f \in C^\infty(M)$, with $0 \leq f \leq 1$, and support of f is contained in U . By faithfulness of ψ on $C^\infty(M)$ we get $0 < \psi(f) = \int_U f d\nu \leq \nu(U)$. \square

3.3 Defining $d\alpha$ for a smooth action α

Let α be a smooth action of a CQG \mathcal{Q} on a manifold M . Recall the $C^\infty(M) \hat{\otimes} \mathcal{Q}$ bimodule $\Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$. However there is also a $C^\infty(M)$ bimodule structure of $\Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ given by

$$f.\Omega := \alpha(f)\Omega, \quad \Omega.f := \Omega\alpha(f),$$

for $\Omega \in \Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ and $f \in C^\infty(M)$ where $\alpha(f)\Omega$ and $\Omega\alpha(f)$ denote the usual left and right $C^\infty(M) \hat{\otimes} \mathcal{Q}$ -bimodule multiplication.

It is easy to identify elements $\Omega \in \Omega^1(C^\infty(M)) \bar{\otimes} \mathcal{Q}$ with \mathcal{Q} -valued smooth one form, i.e.

$\Omega : M \rightarrow \cup_{m \in M} (T_m^* M) \otimes \mathcal{Q}$, such that for all $m \in M$, $\Omega(m) \in T_m^* M \otimes \mathcal{Q}$ and for any coordinate neighborhood U and the local coordinates (x_1, \dots, x_n) around $m \in M$ we can find $\Omega_i \in C^\infty(M) \hat{\otimes} \mathcal{Q}$, $i = 1, \dots, n$, such that $\Omega(x) = \sum_{i=1}^n dx_i(x) \otimes \Omega_i(x)$ for all $x \in U$. We shall usually write $dx_i(x) \otimes \Omega_i(x)$ as $dx_i(x) \Omega_i(x)$ and $\Omega = \sum_{i=1}^n dx_i \Omega_i$ on U .

This allows us to define $\tilde{d} \equiv (d \otimes \text{id})$ from $C^\infty(M) \hat{\otimes} \mathcal{Q}$ to $\Omega \in \Omega^1(C^\infty(M)) \hat{\otimes} \mathcal{Q}$ given by

$$(\tilde{d}F)(m) := \sum_{i=1}^n dx_i(m) \left(\frac{\partial F}{\partial x_i} \right)(m),$$

for $m \in M$ and for any local coordinate chart $(U, (x_1, \dots, x_n))$ around m . Clearly this is uniquely defined by the condition

$$(\text{id} \otimes \omega)(\tilde{d}F) = d((\text{id} \otimes \omega)(F))$$

for all bounded linear functional ω on \mathcal{Q} . Thus $\tilde{d}F$ does not depend on the choice of the local coordinates.

We now have

Theorem 3.8 *The following are equivalent:*

- (i) *There is a well defined, Fréchet continuous map $\beta : \Omega^1(C^\infty(M)) \rightarrow \Omega^1(C^\infty(M)) \hat{\otimes} \mathcal{Q}$ which is a $C^\infty(M)$ bimodule morphism, i.e. $\beta(f\omega) = \alpha(f)\beta(\omega)$ and $\beta(\omega f) = \beta(\omega)\alpha(f)$ for all $\omega \in \Omega^1(C^\infty(M))$ and $f \in C^\infty(M)$ and also $\beta(df) = (d \otimes \text{id})(\alpha(f))$ for all $f \in C^\infty(M)$.*
- (ii) *For all $f, g \in C^\infty(M)$ and all smooth vector fields ν on M ,*

$$(\nu \otimes \text{id})\alpha(f)\alpha(g) = \alpha(g)(\nu \otimes \text{id})\alpha(f) \quad (2)$$

Proof:

Proof of necessity:

We have $\beta(df.g) = (d \otimes \text{id})(\alpha(f))\alpha(g)$, $\beta(g.df) = \alpha(g).(d \otimes \text{id})(\alpha(f))$. But $df.g = g.df$ in $\Omega^1(C^\infty(M))$, which gives $(d \otimes \text{id})(\alpha(f))\alpha(g) = \alpha(g).(d \otimes \text{id})(\alpha(f))$, $\forall f, g \in C^\infty(M)$.

Observe that as ν is a smooth vector field, ν is a Fréchet continuous map from $C^\infty(M)$ to $C^\infty(M)$. Thus it is enough to prove (2) for f, g belonging to the Fréchet dense subalgebra \mathcal{A} as in Theorem 3.4. Let $\alpha(f) = f_{(0)} \otimes f_{(1)}$ and $\alpha(g) = g_{(0)} \otimes g_{(1)}$ (Sweedler's notation). Let $x \in M$ and (U, x_1, \dots, x_n) be a coordinate neighborhood around x . Then $[(d \otimes \text{id})(\alpha(f))\alpha(g)](x) = \sum_{i=1}^n g_{(0)}(x) \frac{\partial f_{(0)}}{\partial x_i}(x) dx_i|_x f_{(1)} g_{(1)}$.

The condition $[(d \otimes \text{id})(\alpha(f))\alpha(g)](x) = [\alpha(g)(d \otimes \text{id})(\alpha(f))](x)$ gives

$$g_{(0)}(x) \frac{\partial f_{(0)}}{\partial x_i}(x) f_{(1)} g_{(1)} = g_{(0)}(x) \frac{\partial f_{(0)}}{\partial x_i}(x) g_{(1)} f_{(1)} \quad (3)$$

for all $i = 1, \dots, n$. Now let $a_i \in C^\infty(M)$ for $i = 1, \dots, n$ such that $\nu(x) = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}|_x$ for all $x \in U$.

So

$$\begin{aligned} & [(\nu \otimes \text{id})(\alpha(f))\alpha(g)](x) \\ &= \sum_{i=1}^n a_i(x) \frac{\partial f_{(0)}}{\partial x_i}(x) g_{(0)}(x) f_{(1)} g_{(1)} \end{aligned}$$

and

$$\begin{aligned} & [\alpha(g)(\nu \otimes \text{id})(\alpha(f))](x) \\ &= \sum_{i=1}^n a_i(x) \frac{\partial f_{(0)}}{\partial x_i}(x) g_{(0)}(x) g_{(1)} f_{(1)} \end{aligned}$$

Hence by (3) $[\alpha(g)(\nu \otimes \text{id})(\alpha(f))](x) = [(\nu \otimes \text{id})(\alpha(f))\alpha(g)](x)$ for all $x \in M$
i.e. $[\alpha(g)(\nu \otimes \text{id})(\alpha(f))] = [(\nu \otimes \text{id})(\alpha(f))\alpha(g)]$ for all $f, g \in \mathcal{A}$.

Proof of sufficiency:

This needs a number of intermediate lemmas. Let $x \in M$ and (U, x_1, \dots, x_n) be a coordinate neighborhood around it. Choose smooth vector fields ν_i 's on M which are $\frac{\partial}{\partial x_i}$ on U . So $[\alpha(g)(\nu_i \otimes \text{id})(\alpha(f))](x) = \frac{\partial f(0)}{\partial x_i}(x)g_{(0)}(x)g_{(1)}f_{(1)}$ and $[(\nu_i \otimes \text{id})(\alpha(f))\alpha(g)](x) = \frac{\partial f(0)}{\partial x_i}(x)g_{(0)}(x)f_{(1)}g_{(1)}$. Hence by the assumption $\sum_i \frac{\partial f(0)}{\partial x_i}(x)g_{(0)}(x)dx_i|_x g_{(1)}f_{(1)} = \sum_i \frac{\partial f(0)}{\partial x_i}(x)g_{(0)}(x)dx_i|_x f_{(1)}g_{(1)}$, hence $[(d \otimes \text{id})(\alpha(f))\alpha(g)](x) = [\alpha(g)(d \otimes \text{id})(\alpha(f))](x)$.

Since x is arbitrary, we conclude that $[\alpha(g)(d \otimes \text{id})(\alpha(f))] = [(d \otimes \text{id})(\alpha(f))\alpha(g)]$ for all $f, g \in \mathcal{A}$. So by Fréchet continuity of d and α we can prove the result for $f, g \in C^\infty(M)$. \square

We use the commutativity to deduce the following:

Lemma 3.9 For $F \in C^\infty(\mathbb{R}^n)$ and $g_1, g_2, \dots, g_n \in C^\infty(M)$

$$(d \otimes \text{id})\alpha(F(g_1, \dots, g_n)) = \sum_{i=1}^n \alpha(\partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)), \quad (4)$$

where $\partial_i F$ denotes the partial derivative of F with respect to the i th coordinate of \mathbb{R}^n .

Proof:

As $\{(g_1(x) \dots g_n(x)) | x \in M\}$ is a compact subset of \mathbb{R}^n , for $F \in C^\infty(\mathbb{R}^n)$, we get a sequence of polynomials P_m in \mathbb{R}^n such that $P_m(g_1, \dots, g_n)$ converges to $F(g_1, \dots, g_n)$ in the Fréchet topology of $C^\infty(M)$.

We see that for P_m ,

$$\begin{aligned} & (d \otimes \text{id})(\alpha(P_m(g_1, \dots, g_n))) \\ &= (d \otimes \text{id})(P_m(\alpha(g_1, \dots, g_n))) \\ &= \sum_{i=1}^n \alpha(\partial_i P_m(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)), \end{aligned}$$

using $(d \otimes \text{id})(\alpha(f))\alpha(g) = \alpha(g)(d \otimes \text{id})\alpha(f)$ as well as the Leibniz rule for $(d \otimes \text{id})$.

The lemma now follows from Fréchet continuity of α and $(d \otimes \text{id})$. \square

Lemma 3.10 Let U be a coordinate neighborhood. Also let $g_1, g_2, \dots, g_n \in C^\infty(M)$ be such that $(g_1|_U, \dots, g_n|_U)$ gives a local coordinate system on U . Then

$$(d \otimes \text{id})(\alpha(f)) = \sum_{j=1}^n \alpha(\partial_{g_j} f)(d \otimes \text{id})(\alpha(g_j)),$$

for all $f \in C^\infty(M)$ supported in U .

Proof:

Let $F \in C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a smooth function such that $f(m) = F(g_1(m), \dots, g_n(m)) \forall m \in U$. Choose $\chi \in C^\infty(M)$ with $\chi \equiv 1$ on $K = \text{supp}(f)$ and $\text{supp}(\chi) \subset U$. Then $\chi f = f$ as $\chi \equiv 1$ on K . Hence $\chi F(g_1, \dots, g_n) = f(\chi F = \chi f = f \text{ on } U, \chi F = 0 \text{ outside } U)$. Also $\chi^2 F(g_1, \dots, g_n) = \chi F(g_1, \dots, g_n)$, since on K , $\chi^2 = \chi = 1$ and outside K , $\chi^2 F(g_1, \dots, g_n) = \chi F(g_1, \dots, g_n) = 0$. Let $T := \alpha(\chi)$ and $S := \alpha(F(g_1, \dots, g_n))$. Also denote $(d \otimes \text{id})\alpha(F(g_1, \dots, g_n))$ by S' and $(d \otimes \text{id})(\alpha(\chi))$ by T' .

So we have $T^2 S = T S$ and by (2) we have $T' T = T T'$ and $S' S = S S'$.

$$\begin{aligned}
T^2 S' &= \alpha(\chi^2)(d \otimes \text{id})(\alpha(F(g_1, \dots, g_n))) \\
&= \alpha(\chi^2) \sum_{i=1}^n \alpha(\partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \text{ (by (4))} \\
&= \alpha(\chi) \sum_{i=1}^n \alpha(\chi \partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \\
&= \alpha(\chi) \sum_{i=1}^n \alpha(\partial_{g_i} f)(d \otimes \text{id})(\alpha(g_i)) \text{ (as } \text{supp}(\partial_{g_i} f) \subset K). \tag{5}
\end{aligned}$$

$$\begin{aligned}
TS' &= \alpha(\chi)(d \otimes \text{id})(\alpha(F(g_1, \dots, g_n))) \\
&= \sum_{i=1}^n \alpha(\chi \partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \\
&= \sum_{i=1}^n \alpha(\chi^2 \partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \\
&= \alpha(\chi) \sum_{i=1}^n \alpha(\partial_{g_i} f)(d \otimes \text{id})(\alpha(g_i)) \tag{6}
\end{aligned}$$

Combining (5) and (6) we get

$$T^2 S' = TS' \tag{7}$$

Now

$$\begin{aligned}
T^2 S &= TS \\
\Rightarrow (d \otimes \text{id})(T^2 S) &= (d \otimes \text{id})(TS) \\
\Rightarrow 2TT'S + T^2 S' &= TS' + T'S \text{ (by Leibniz rule and } T'T = TT') \\
\Rightarrow 2TT'S &= T'S \text{ (by (7))} \\
\Rightarrow 2\alpha(\chi)(d \otimes \text{id})(\alpha(\chi))\alpha(F(g_1, \dots, g_n)) &= (d \otimes \text{id})(\alpha(\chi))\alpha(F(g_1, \dots, g_n)) \\
\Rightarrow 2\alpha(\chi^2)(d \otimes \text{id})(\alpha(\chi))\alpha(F(g_1, \dots, g_n)) &= \alpha(\chi)(d \otimes \text{id})(\alpha(\chi))\alpha(F(g_1, \dots, g_n)) \\
\Rightarrow 2(d \otimes \text{id})(\alpha(\chi))\alpha(f) &= (d \otimes \text{id})(\alpha(\chi))\alpha(f) \text{ (using the assumption and } \chi^2 F = f) \\
\Rightarrow (d \otimes \text{id})(\alpha(\chi))\alpha(f) &= 0 \tag{8}
\end{aligned}$$

So

$$\begin{aligned}
(d \otimes \text{id})(\alpha(f)) &= (d \otimes \text{id})(\alpha(\chi f)) \\
&= (d \otimes \text{id})(\alpha(\chi))\alpha(f) + \alpha(\chi)(d \otimes \text{id})(\alpha(f)) \\
&= \alpha(\chi)(d \otimes \text{id})(\alpha(f)) \text{ (by (8))} \\
&= \alpha(\chi)(d \otimes \text{id})(\alpha(\chi F(g_1, \dots, g_n))) \\
&= \alpha(\chi)(d \otimes \text{id})(\alpha(\chi))\alpha(F(g_1, \dots, g_n)) + \alpha(\chi^2)(d \otimes \text{id})(\alpha(F(g_1, \dots, g_n))) \\
&= (d \otimes \text{id})(\alpha(\chi))\alpha(f) + \alpha(\chi^2)(d \otimes \text{id})(\alpha(F(g_1, \dots, g_n))) \text{ (Again by assumption)} \\
&= \alpha(\chi^2) \sum_{i=1}^n \alpha(\partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \text{ (by (4) and (8))} \\
&= \sum_{i=1}^n \alpha(\chi^2 \partial_i F(g_1, \dots, g_n))(d \otimes \text{id})(\alpha(g_i)) \\
&= \sum_{i=1}^n \alpha(\partial_{g_i} f)(d \otimes \text{id})(\alpha(g_i))
\end{aligned}$$

□

Now to complete the proof of the theorem, we want to first define a bimodule morphism β extending $d\alpha$ locally, i.e. we define $\beta_U(\omega)$ for any coordinate neighborhood U and any smooth 1-form ω supported in U as follows:

Choose C^∞ functions $g_1 \dots g_n$ as before such that they give a local coordinate system on U and ω has the unique expression $\omega = \sum_{j=1}^n \phi_j dg_j$. Then define $\beta_U(\omega) := \sum_{j=1}^n \alpha(\phi_j)(d \otimes \text{id})\alpha(g_j)$. We claim that β_U is independent of the choice of the coordinate functions (g_1, \dots, g_n) , i.e. if (h_1, \dots, h_n) is another such set of coordinate functions on U with $\omega = \sum_{j=1}^n \psi_j dh_j$ for some ψ_j 's in $C^\infty(M)$, then

$$\sum_{j=1}^n \alpha(\phi_j)(d \otimes \text{id})(\alpha(g_j)) = \sum_{j=1}^n \alpha(\psi_j)(d \otimes \text{id})(\alpha(h_j)).$$

Indeed let χ be a smooth function which is 1 on the support of ω and 0 outside U . We have $F_1, \dots, F_n \in C^\infty(\mathbb{R}^N)$ such that $g_j = F_j(h_1, \dots, h_n)$ for all $j = 1, \dots, n$ on U . Then $\chi g_j = \chi F_j(h_1, \dots, h_n)$ for all $j = 1, \dots, n$. Hence $dg_j = \sum_{k=1}^n \partial_{h_k}(F_j(h_1, \dots, h_n))dh_k$ on U . That is $\omega = \sum_{j,k} \chi \phi_j \partial_{h_k}(F_j(h_1, \dots, h_n))dh_k$. So $\psi_k = \sum_j \chi \phi_j \partial_{h_k}(F_j(h_1, \dots, h_n))$. Also, note that, as $\chi \equiv 1$ on the support of ϕ_j for all j , we must have $\phi_j \partial_{h_k}(\chi) \equiv 0$, so $\chi \phi_j \partial_{h_k}(F_j(h_1, \dots, h_n)) = \chi \phi_j \partial_{h_k}(\chi F_j(h_1, \dots, h_n))$. Thus

$$\begin{aligned} & \sum_k \alpha(\psi_k)(d \otimes \text{id})(\alpha(h_k)) \\ &= \sum_{k,j} \alpha(\chi \phi_j \partial_{h_k}(F_j(h_1, \dots, h_n)))(d \otimes \text{id})(\alpha(h_k)) \\ &= \sum_{k,j} \alpha(\phi_j) \alpha(\partial_{h_k}(\chi F_j(h_1, \dots, h_n)))(d \otimes \text{id})(\alpha(h_k)) \\ &= \sum_j \alpha(\phi_j)(d \otimes \text{id})(\alpha(\chi F_j(h_1, \dots, h_n))) \text{ (by Lemma 3.9)} \\ &= \sum_j \alpha(\phi_j)(d \otimes \text{id})(\alpha(\chi g_j)) \\ &= \sum_j \alpha(\phi_j)(d \otimes \text{id})(\alpha(g_j)) \end{aligned}$$

Where the last step follows from Leibniz rule and the fact that

$$\begin{aligned} & \alpha(\phi_j)(d \otimes \text{id})(\alpha(\chi)) \\ &= \sum_k \alpha(\phi_j) \alpha(\partial_{h_k}(\chi))(d \otimes \text{id})(\alpha(h_k)) \\ &= \sum_k \alpha(\phi_j \partial_{h_k}(\chi))(d \otimes \text{id})(\alpha(h_k)) \\ &= 0 \text{ (using } \phi_j \partial_{h_k}(\chi) \equiv 0), \end{aligned}$$

which proves the claim.

Hence the definition is indeed independent of choice of coordinate system. Then for any two coordinate neighborhoods U and V , $\beta_U(\omega) = \beta_V(\omega)$ for any ω supported in $U \cap V$. It also follows from the definition and the given condition (2) that β_U is a $C^\infty(M)$ -bimodule morphism. Moreover we get from Lemma 3.10 that $\beta_U(df) = (d \otimes \text{id})\alpha(f)$ for all $f \in C^\infty(M)$ supported in U . Now we define β globally as follows:

Choose (and fix) a smooth partition of unity $\{\chi_1, \dots, \chi_l\}$ subordinate to a cover $\{U_1, \dots, U_l\}$

of the manifold M such that each U_i is a coordinate neighborhood. Define β by:

$$\beta(\omega) := \sum_{i=1}^l \beta_{U_i}(\chi_i \omega),$$

for any smooth one form ω . Then for any $f \in C^\infty(M)$,

$$\begin{aligned} \beta(df) &= \sum_{i=1}^l \beta_{U_i}(\chi_i df) \\ &= \sum_{i=1}^l \beta_{U_i}(d(\chi_i f) - f d\chi_i) \\ &= \sum_{i=1}^l [(d \otimes \text{id})(\alpha(\chi_i f)) - \alpha(f)(d \otimes \text{id})(\alpha(\chi_i))] \\ &= \sum_{i=1}^l \alpha(\chi_i)(d \otimes \text{id})(\alpha(f)) \text{ (by Leibniz rule)} \\ &= (d \otimes \text{id})(\alpha(f)) \end{aligned}$$

This completes the proof of the Theorem 3.8.

□

We end this subsection with an example of Hopf-algebra (of non compact type) having coaction on a coordinate algebra of an algebraic variety which violates the condition (2). However as mentioned in the introduction, we don't have any example of a smooth CQG action which violates the condition (2).

Let $\mathcal{A} \equiv \mathbb{C}[x]$ be the $*$ algebra of polynomials in one variable with complex coefficients and \mathcal{Q}_0 be the Hopf $*$ algebra generated by a, a^{-1}, b subject to the following relations

$$aa^{-1} = a^{-1}a = I, ab = q^2ba,$$

where q is a parameter as described in [21]. This Hopf algebra corresponds to the quantum $ax + b$ group. There are at least two different (non-isomorphic) constructions of the analytic versions of this quantum group i.e. as locally compact quantum groups in the sense of [9], one by Woronowicz ([21]) the other by Baaaj-Skandalis ([1]) and Vaes-Veineremann ([18]). The coproduct is given by (see [21] for details)

$$\Delta(a) = a \otimes a, \Delta(b) = a \otimes b + b \otimes I.$$

We have a coaction $\alpha : k[x] \rightarrow k[x] \otimes \mathcal{Q}_0$ given by

$$\alpha(x) = x \otimes a + 1 \otimes b.$$

The algebra \mathcal{A} is the algebraic geometric analogue of $C^\infty(\mathbb{R})$ and we have the following canonical derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$ corresponding to the vector field $\frac{d}{dt}$ of \mathbb{R} :

$$\delta(p) = p',$$

where p' denotes the usual derivative of the polynomial p . However an easy computation gives

$$(\delta \otimes \text{id})\alpha(x) = 1 \otimes a,$$

which do not commute with $\alpha(x)$ as $ab \neq ba$.

4 Actions which preserve some Riemannian inner product

Definition 4.1 Suppose that M has a Riemannian structure with the corresponding $C^\infty(M)$ valued inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\Omega^1(C^\infty(M))$. We call a smooth action α on M to be inner product preserving on a Fréchet-dense unital $*$ -subalgebra \mathcal{A} of $C^\infty(M)$ if

$$\langle\langle (d \otimes \text{id})\alpha(f), (d \otimes \text{id})\alpha(g) \rangle\rangle = \alpha(\langle\langle df, dg \rangle\rangle) \quad (9)$$

for all $f, g \in \mathcal{A}$. When $\mathcal{A} = C^\infty(M)$ we simply call the smooth action simply inner product preserving.

Given a smooth action α of a CQG \mathcal{Q} on a Riemannian manifold M , it is easy to see, by Fréchet continuity of the maps d and α that the equation (9) with f, g varying in any Fréchet dense $*$ -subalgebra of $C^\infty(M)$ is equivalent to having it for all $f, g \in C^\infty(M)$.

Theorem 4.2 If α is a smooth action of a CQG \mathcal{Q} on a compact, smooth manifold. Then it admits a lift $d\alpha$ as a well defined α -equivariant unitary representation on the Hilbert bimodule of one forms satisfying $d\alpha(df) = (d \otimes \text{id})\alpha(f)$ if and only if there is a Riemannian structure on the manifold such that α is inner product preserving.

Proof:

First we suppose that α is inner product preserving with respect to some Riemannian inner product. Let \mathcal{A} be the maximal $*$ -subalgebra of $C^\infty(M)$ over which the action is algebraic. Then by the remark after definition (4.1), we see that α is inner product preserving on \mathcal{A} . For $\omega = \sum_i f_i dg_i$, for $f_i, g_i \in \mathcal{A}$ we define $d\alpha(\omega) := \sum_i (d \otimes \text{id})\alpha(g_i)\alpha(f_i)$. Also let $\eta = \sum_i f_{i'} dg_{i'}$, where $f_{i'}, g_{i'} \in \mathcal{A}$, then $\langle\langle d\alpha(\omega), d\alpha(\eta) \rangle\rangle = \alpha(\langle\langle \omega, \eta \rangle\rangle)$. Then it is a well defined α equivariant bimodule morphism. The coassociativity condition follows from that of α . Moreover, as $\text{Sp } \alpha(\mathcal{A})(1 \otimes \mathcal{Q}_0) = \mathcal{A} \otimes \mathcal{Q}_0$, we have $\text{Sp } d\alpha(\Omega^1(\mathcal{A}))(1 \otimes \mathcal{Q}_0) = \Omega^1(\mathcal{A}) \otimes \mathcal{Q}_0$.

Now we prove the converse. This is actually an adaptation of the proof of Theorem 3.1 of [6]. In this case we no longer need the assumption I of that theorem thanks to Lemma 3.5 (also see [7]). We can assume without loss of generality that the quantum group is reduced, i.e. $\mathcal{Q} = \mathcal{Q}_r$, hence the Haar state h is faithful and the antipode is bounded. To justify this, observe that as α is algebraic on \mathcal{A} , the corresponding reduced action α_r (say) of \mathcal{Q}_r given by $(\text{id} \otimes \pi_r) \circ \alpha$ (where π_r denotes the quotient morphism from \mathcal{Q} onto \mathcal{Q}_r) coincides with α on \mathcal{A} . For this reason, it is enough to show that the reduced action α_r is inner product preserving for some Riemannian structure on \mathcal{A} . In other words, we can replace \mathcal{Q} by \mathcal{Q}_r in this proof, and as we already know that \mathcal{Q}_r must be a Kac algebra, antipode is norm-bounded on it. This proves the claim.

Now, we briefly give the construction of a Riemannian metric which will be preserved by the action, following the lines of arguments in [6]. Basically, we have to note that one does not need Assumption I of that paper. Choose and fix a Riemannian inner product on M , with $\langle\langle \cdot, \cdot \rangle\rangle$ denoting the corresponding $C^\infty(M)$ -valued inner product on one-forms and as before, consider the Fréchet dense $*$ -algebra \mathcal{A} on which α is algebraic. However, unlike [6], $\langle\langle df, dg \rangle\rangle$ may not belong to \mathcal{A} for $f, g \in \mathcal{A}$. However, we'll see the arguments of [6] still work. Extend the definition of the map Ψ defined in Lemma 3.3 of that paper to $C^\infty(M) \otimes \mathcal{Q}_0$ (algebraic tensor product), i.e. let $\Psi(F)$ in $C^\infty(M)$ be given by

$$\Psi(F)(x) := h(m \circ (\kappa \otimes \text{id})(G)),$$

where $G = (\alpha \otimes \text{id})(F)(x)$ and where m, κ etc. are as in that paper, h the (faithful) Haar state of \mathcal{A} . Note that m is well-defined on $\mathcal{Q} \otimes \mathcal{Q}_0$, so there is no problem in the above definition. At this point, let us make a useful observation: in the definition of Ψ , we may put κ on the rightmost position, i.e. we have $\Psi(F)(x) = \Psi^\sharp(F)(x) := h(m \circ (\text{id} \otimes \kappa)(G))$. By continuity of

the maps involved, it suffices to verify this for $F = f \otimes q$, where $f \in \mathcal{A}, q \in \mathcal{Q}_0$, and in this case, we have the following using $\kappa^2 = \text{id}$, $h \circ \kappa = h$ and the traciality of h :

$$\Psi(F) = f_{(0)}h(\kappa(f_{(1)})q) = f_{(0)}h(\kappa(q)f_{(1)}) = f_{(0)}h(f_{(1)}\kappa(q)) = \Psi^\sharp(F).$$

The proof of complete positivity of Ψ as in [6] goes through for the extended map without any change in the arguments, i.e. the conclusion of Lemma 3.3 of that paper holds. For $\omega, \eta \in \Omega^1(\mathcal{A})$

We define

$$<< \omega, \eta >>' := \Psi(<< d\alpha(\omega), d\alpha(\eta) >>)$$

and observe that the proof of Lemma 3.4, Lemma 3.5 and Lemma 3.6 of [6] go through almost verbatim in our case. In fact, the only care necessary is about $x = << d\phi_{(0)}, d\psi_{(0)} >>$ (in the notation of [6]) which no longer belongs to \mathcal{A} , so writing Sweedler-type notation for $\alpha(x)$ is not permitted. However, we can approximate x by a sequence of elements x_n in \mathcal{A} and do calculations similar to [6] to get the desired conclusions of Lemma 3.4, 3.5 and 3.6. This shows $< \cdot, \cdot >'$ is a nonnegative definite sesqui-linear form. Let us give some details of the proof of strict positive definiteness of $< \cdot, \cdot >'$. Let v and s_1, \dots, s_n be as in the proof of Theorem 3.1 of [6] and suppose $< v, v >' = 0$ i.e.

$$\sum_{i,j} \bar{c}_i c_j << ds_i, ds_j >>'(x) = 0,$$

where $v = \sum_i c_i ds_i(x) \in T_x^*(M)$. By faithfulness of h and also using the observation $\Psi = \Psi^\sharp$, which allows to replace $\kappa \otimes \text{id}$ by $\text{id} \otimes \kappa$, we get the following:

$$\sum_{i,j} \bar{c}_i c_j ((\text{id} \otimes m)(\text{id} \otimes \text{id} \otimes \kappa)(\alpha \otimes \text{id}) << d\alpha(ds_i), d\alpha(ds_j) >>)(x) = 0,$$

i.e.

$$\sum_{i,j} \bar{c}_i c_j \alpha(<< ds_{i(0)}, ds_{j(0)} >>)(x) \kappa(s_{i(1)}^* s_{j(1)}) = 0.$$

Applying the extension of ϵ constructed in Corollary 3.6 to the above equation and using $\epsilon \circ \kappa = \epsilon$ on \mathcal{Q}_0 , we get

$$\sum_{i,j} \bar{c}_i c_j << ds_{i(0)}, ds_{j(0)} >> (x) \epsilon(s_{i(1)}^* s_{j(1)}) = 0. \quad (10)$$

But for $f, g \in \mathcal{A}$, $<< df_{(0)}, dg_{(0)} >> \epsilon(f_{(1)}^* g_{(1)}) = << df, dg >>$, using $f_{(0)}\epsilon(f_{(1)}) = f$ for all $f \in \mathcal{A}$. Thus, Using the fact that ϵ is $*$ -homomorphism, (10) reduces to

$$< \sum_i c_i ds_i(x), \sum_i c_i ds_i(x) > = 0,$$

i.e $< v, v > = 0$. \square

5 Appendix: Proof of

$$C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B} \cong C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$$

Let M be a smooth, compact n -dimensional manifold and \mathcal{A}, \mathcal{B} be two C^* algebras. For any nice algebra \mathcal{C} , let $\text{Der}(\mathcal{C})$ denote the set of its all closable derivations. From the definition of topological tensor product of two nice algebras in our sense, and as \mathcal{B} is a C^* algebra, $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B}$ is the completion of $C^\infty(M, \mathcal{A}) \otimes \mathcal{B}$ with respect to the family of seminorms given by the family of closable $*$ derivations of the form $\{\tilde{\eta} := \eta \otimes \text{id} : \eta \in \text{Der}(C^\infty(M, \mathcal{A}))\}$, where η is any (closable $*$)-derivation on $C^\infty(M, \mathcal{A})$. On the other hand, we have the natural Fréchet topology on $C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$ coming from the derivations of the form $\delta \otimes \text{id}_{\mathcal{A} \hat{\otimes} \mathcal{B}}$, where δ is any (closable $*$) derivation on $C^\infty(M)$, i.e. a smooth vector field. We want to show that these two topologies coincide.

Lemma 5.1 Consider $C^\infty(M, \mathcal{A})$ as a $C^\infty(M)$ bimodule using the algebra inclusion $C^\infty(M) \cong C^\infty(M) \otimes 1 \subset C^\infty(M, \mathcal{A})$. Let $D : C^\infty(M) \mapsto C^\infty(M, \mathcal{A})$ be a derivation. Then given any coordinate neighborhood (U, x_1, \dots, x_n) , there exist $a_1, \dots, a_n \in C^\infty(M, \mathcal{A})$ such that for any $m \in U$,

$$D(f)(m) = \sum_{i=1}^n a_i(m) \frac{\partial f}{\partial x_i}(m).$$

Proof

It follows by standard arguments similar to those used in proving that any derivation on $C^\infty(M)$ is a vector field. \square

Corollary 5.2 Let η be a closable $*$ -derivation on $C^\infty(M, \mathcal{A})$. Then there exists a norm bounded $*$ -derivation $\eta^A : \mathcal{A} \rightarrow C^\infty(M, \mathcal{A})$ and finitely many smooth vector fields ξ_{ij} , elements $a_{ij} \in C^\infty(M, \mathcal{A})$, $i = 1, \dots, n$, $j = 1, \dots, p$ (p positive integer) such that

$$\eta(F)(m) = \sum_{i,j} a_{ij}(m) (\xi_{ij} \otimes \text{id})(F)(m) + \eta^A(F(m))(m). \quad (11)$$

Proof:

Choose any finite cover U_1, \dots, U_p by coordinate neighborhoods and let χ_1, \dots, χ_p be the associated smooth partition of unity. Define $\eta^A(q) := \eta(1 \otimes q)$. As any closed $*$ derivation on a C^* algebra is norm bounded, we get from Lemma 5.1 and the observation that $\eta(f \otimes q) = \eta(f \otimes 1)(1 \otimes q) + (f \otimes 1)\eta(1 \otimes q)$ the following expression of $\eta(F)(m)$ for m in a coordinate neighborhood U_j , say, where $(x_1^{(j)}, \dots, x_n^{(j)})$ are the corresponding local coordinates:

$$(\eta F)(m) = \sum_{i=1}^n a_{ij}(m) \frac{\partial F}{\partial x_i^{(j)}}(m) + \eta^A(F(m))(m),$$

where $a_{ij} \in C^\infty(M, \mathcal{A})$. Now, the lemma follows by taking $\xi_{ij} = \chi_j \frac{\partial}{\partial x_i^{(j)}}$. \square

Corollary 5.3 $C^\infty(M, \mathcal{A})$ is a nice algebra.

Proof:

Denote by τ the topology on $C^\infty(M, \mathcal{A})$ coming from all closable $*$ -derivations. As the usual Fréchet topology on this space is given by derivations of the form $\delta \otimes \text{id}$ where δ is a smooth vector field on M , clearly τ is stronger than the usual topology. Let us show the other direction. The expression of any $\eta \in \text{Der}(C^\infty(M, \mathcal{A}))$ given any (26) of Corollary 5.2 ensures that η is continuous w.r.t. the usual Fréchet topology of $C^\infty(M, \mathcal{Q})$. Thus, a sequence F_n of $C^\infty(M, \mathcal{A})$ which is Cauchy in the usual topology of $C^\infty(M, \mathcal{A})$ will be Cauchy in the τ -topology too. It follows that τ is weaker than the usual Fréchet topology of $C^\infty(M, \mathcal{A})$, hence the two topologies coincide. \square

Lemma 5.4 Let $F \in C(M, \mathcal{A} \hat{\otimes} \mathcal{B})$ such that for all $\omega \in \mathcal{B}^*$, where \mathcal{B}^* denotes the space of all bounded linear functionals on \mathcal{B} , $(\text{id} \otimes \text{id} \otimes \omega)F \in C^\infty(M, \mathcal{A})$. Then $F \in C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$.

Proof:

We first prove it when M is an open subset U of \mathbb{R}^n with compact closure (say) K . We denote the standard coordinates of \mathbb{R}^n by $\{x_1, \dots, x_n\}$. Let us choose a point $x^0 = (x_1^0, \dots, x_n^0)$ on the manifold and $h, h' > 0$ such that $(x_1^0, \dots, x_i^0 + h, \dots, x_n^0)$ and $(x_1^0, \dots, x_i^0 + h', \dots, x_n^0)$ both belong

to the open set U for a fixed $i \in \{1, \dots, n\}$. We shall show that $\frac{\partial F}{\partial x_i}(x^0)$ exists. That is, we have to show that

$$\Omega^F(x^0; h) := \frac{F(x_1^0, \dots, x_i^0 + h, \dots, x_n^0) - F(x_1^0, \dots, x_i^0, \dots, x_n^0)}{h}$$

is Cauchy in $\mathcal{A} \hat{\otimes} \mathcal{B}$ as $h \rightarrow 0$. For that first observe that $((\text{id} \otimes \text{id} \otimes \omega)F)(x) = (\text{id} \otimes \omega)(F(x))$ for all $x \in M$ and $\omega \in \mathcal{B}^*$. Now

$$\begin{aligned} & (\text{id} \otimes \omega)(\Omega^F(x^0; h) - \Omega^F(x^0; h')) \\ = & \frac{h'((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0 + h, \dots, x_n^0) - ((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0, \dots, x_n^0)}{hh'} \\ & - \frac{h((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0 + h', \dots, x_n^0) - ((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0, \dots, x_n^0)}{hh'} \\ = & \frac{h' \int_0^h \frac{\partial}{\partial x_i}((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0 + u, \dots, x_n^0) du}{hh'} \\ & - \frac{h \int_0^{h'} \frac{\partial}{\partial x_i}((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0 + v, \dots, x_n^0) dv}{hh'} \\ = & \frac{\int_0^h \int_0^{h'} du dv \int_v^u \frac{\partial^2}{\partial x_i^2}((\text{id} \otimes \text{id} \otimes \omega)F)(x_1^0, \dots, x_i^0 + s, \dots, x_n^0) ds}{hh'}, \end{aligned}$$

where all the integrals involved above are Banach space valued Bochner integrals. Let $\sup_{x \in K} \|\frac{\partial^2}{\partial x_i^2}((\text{id} \otimes \text{id} \otimes \omega)F)(x)\| = M_\omega$. Then using the fact that for a regular Borel measure μ and a Banach space valued function F , $\|\int F d\mu\| \leq \int \|F\| d\mu$, we get

$$\|(\text{id} \otimes \omega)(\Omega^F(x^0; h) - \Omega^F(x^0; h'))\| \leq M_\omega \epsilon,$$

where $\epsilon = \min\{h, h'\}$. Now consider the family $\beta_{x^0; h, h'}^\phi = (\phi \otimes \text{id})(\Omega^F(x^0; h) - \Omega^F(x^0; h'))$ for $\phi \in \mathcal{A}^*$ with $\|\phi\| \leq 1$. For $\omega \in \mathcal{B}^*$,

$$\omega(\beta_{x^0; h, h'}^\phi) = (\phi \otimes \text{id})(\text{id} \otimes \omega)(\Omega^F(x^0; h) - \Omega^F(x^0; h')).$$

Hence we have

$|\omega(\beta_{x^0; h, h'}^\phi)| \leq \|(\text{id} \otimes \omega)(\Omega^F(x^0; h) - \Omega^F(x^0; h'))\| \leq M_\omega \epsilon$. By the uniform boundedness principle we get a constant $M > 0$ such that $\|(\beta_{x^0; h, h'}^\phi)\| \leq M \epsilon$. But $\|(\Omega^F(x^0; h) - \Omega^F(x^0; h'))\| = \sup_{\|\phi\| \leq 1} \|\beta_{x^0; h, h'}^\phi\|$. Therefore we get

$$\|(\Omega^F(x^0; h) - \Omega^F(x^0; h'))\| \leq M \epsilon \text{ for all } h, h'.$$

Hence $\Omega^F(x^0; h)$ is Cauchy as h goes to zero i.e. $\frac{\partial F}{\partial x_i}(x^0)$ exists. By similar arguments we can show the existence of higher order partial derivatives. For a general smooth, compact manifold M , going to the coordinate neighborhood and applying the above result we can show that $F \in C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$. \square

Applying the above Lemma for $\mathcal{A} = \mathbb{C}$, we get

Corollary 5.5 *For $f \in C(M, \mathcal{B})$, if $(\text{id} \otimes \phi)f \in C^\infty(M)$ for all $\phi \in \mathcal{B}^*$, then $f \in C^\infty(M, \mathcal{B})$.*

Now we are ready to prove the main result of this appendix.

Lemma 5.6 *We have the following isomorphism of Fréchet $*$ -algebras:*

$$C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B} \cong C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B}).$$

Proof:

First we show that

$$C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B}) \subseteq C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B},$$

and the inclusion map is Fréchet continuous. To prove the above inclusion it is enough to show that if a sequence in $C^\infty(M) \otimes \mathcal{A} \otimes \mathcal{B}$ is Cauchy in the topology of the L.H.S., it is also Cauchy in the topology of the R.H.S.. This follows from the descriptions of derivations on the algebra $C^\infty(M, \mathcal{A})$ given in the Lemma 5.2 and the fact that, \mathcal{B} being a C^* algebra, the topology on the right hand side is given by the derivations $\eta \otimes \text{id}$'s, $\eta \in \text{Der}(C^\infty(M))$ where η 's are (closable $*$) derivations on $C^\infty(M, \mathcal{A})$.

Moreover, observe that for any $\omega \in \mathcal{B}^*$ and $F \in C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B}$, $(\text{id} \otimes \omega)F \in C^\infty(M, \mathcal{A})$. Hence by Lemma 5.4, we get $C^\infty(M, \mathcal{A}) \hat{\otimes} \mathcal{B} \subseteq C^\infty(M, \mathcal{A} \hat{\otimes} \mathcal{B})$ as well, i.e. the two spaces coincide as sets. By the closed graph theorem we conclude that they are isomorphic as Fréchet spaces. \square

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